GEOS F493 / F693
Geodetic Methods and Modeling

– Lecture 03a: Linear Algebra Review–

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Scalar vs. Vector

- Scalar:

- Vector:

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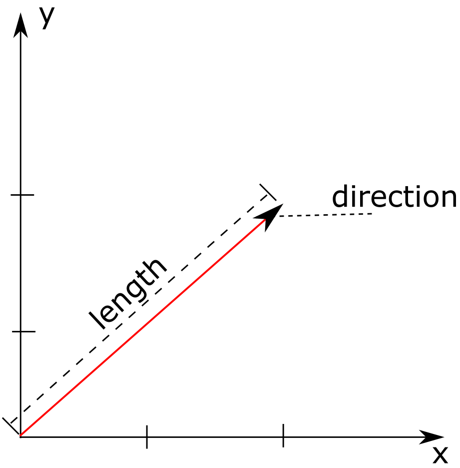

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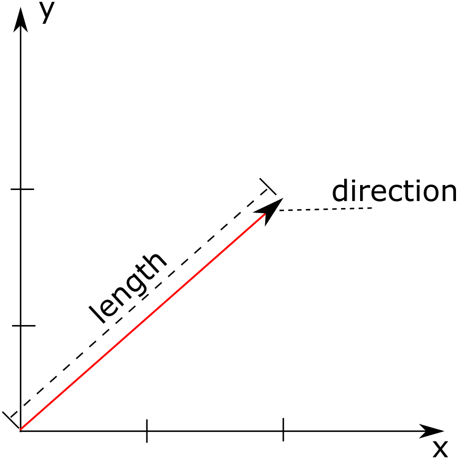
- Scalar: a number, provides a magnitude
- Vector: has magnitude **and** direction

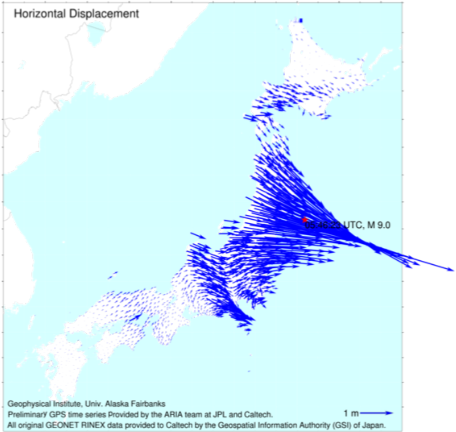
Examples:

- Scalar: speed, length
- Vector: velocity, displacement

$$
\begin{aligned}
\vec{v} &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
\vec{u} &= \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
\vec{a} &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}
\end{aligned}
$$



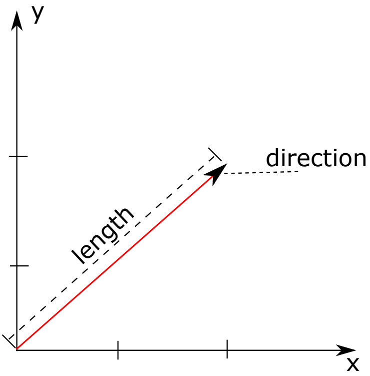
Vector Field



Vectors

What is its length? (Vector Norm)

$$
\begin{aligned}
||\vec{v}|| &= \sqrt{(v_1^2 + v_2^2)} \\
||\vec{v}|| &= \sqrt{(v^T \cdot v)} \\
&= \sqrt{(v_1 v_1 + v_2 v_2 + \cdots v_n v + n)}
\end{aligned}
$$



Dot Product in General

$$
\begin{aligned}
\vec{v} \cdot \vec{u} &= \vec{v}^T \vec{u} \\
&= \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \\
&= c
\end{aligned}
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where c is a scalar!

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$$\vec{v} \cdot \vec{u} = ||\vec{v}||||\vec{u}||cos(\theta)$$

where θ is the angle between $\vec{v}, \vec{u}$.

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What if $\theta = 90^\circ$?
$cos(\theta) = 0$ . . . use dot product to test for orthogonality!

Linear Combinations

We can apply arithmetic operations on vectors:

$$c \cdot \vec{v} + d \cdot \vec{u} \quad = \quad \vec{y}$$

is a linear combination of $\vec{v}$ and $\vec{u}$ using weights / coefficients c, d to combine vector elements.

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Example:
Let

$$\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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Systems of Linear Equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned}$$

we want to solve for x, y, z. We have 3 equations and 3 unknowns, usually giving 1 solution. Other systems may not have a solution.

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\mathbf{Gm} &= \mathbf{d}
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$$

in this general form **G** is a matrix m rows and n columns ($m \times n$ matrix):

$$
\mathbf{G} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{bmatrix}
$$

$\mathbf{Gm} = \mathbf{d}$ only has a solution if **d** is a linear combination of the columns in **G**.

Matrix Operations

If $\mathbf{G} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{bmatrix}$ then

$$3\mathbf{G} = \begin{bmatrix} 3g_{11} & 3g_{12} & \cdots & 3g_{1n} \\ 3g_{21} & 3g_{22} & \cdots & 3g_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ 3g_{m1} & 3g_{m2} & \cdots & 3g_{mn} \end{bmatrix}$$

Matrix addition is element-wise (same as vector addition), requires same dimensions for both matrices.

Matrix Operations

Matrix multiplication:
A $\cdot$ **B** or **AB** is defined if the number of columns in **A** are the same as the number of rows in **B**.

If **A** is (m x n), then **B** must be (n x m), inner dimensions must agree.

Example on board.

Matrix Transpose

If **A** is (m x n) matrix, then $\mathbf{A}^T$ the transpose of **A** is (n x m) matrix with columns made up of the rows of **A**:

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$\mathbf{A}^T =$$

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\mathbf{A} &= \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \\
\mathbf{A}^T &= \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}
\end{aligned}
$$

Matrix Transpose

Some properties:

$$\begin{aligned}(\mathbf{A}^T)^T &= \mathbf{A}\\(\mathbf{A}+\mathbf{B})^T &= \mathbf{A}^T+\mathbf{B}^T\\(\mathbf{AB})^T &= \mathbf{B}^T\mathbf{A}^T\end{aligned}$$

Some additional terms:
Diagonal Matrix: Off-diagonal entries are zero
Identity Matrix: diagonal contains ones, rest zero
Matrix Trace: Sum of the diagonal elements

Transformation Example

On Board.

Linear Independence

The vectors $\vec{z}_1, \vec{z}_2, \cdots, \vec{z}_n$ are **linearly independent** if the system of equations:

$$c_1\vec{z}_1 + c_2\vec{z}_2 + \cdots + c_n\vec{z}_n = \vec{0} \tag{1}$$

has only the trivial solution $\vec{c} = 0$. If there are multiple solutions, then the vectors are **linearly dependent**.

Matrix Inverse

Often we want to solve $\mathbf{Gm} = \mathbf{d}$ not for $\mathbf{d}$, but for $\mathbf{m}$!

Think equation $4x = 3 \ldots$ What's x?

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Yes, $x = 3/4$!

For matrices:

$$\begin{aligned} \mathbf{Gm} &= \mathbf{d} \\ \mathbf{G}^{-1}\mathbf{Gm} &= \mathbf{G}^{-1}\mathbf{d} \\ \mathbf{m} &= \mathbf{G}^{-1}\mathbf{d} \end{aligned}$$

How to find $\mathbf{G}^{-1}$?

Matrix Inverse

How to find $\mathbf{G}^{-1}$?

We can try the determinant (example for 2 x 2 matrix):

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $det(\mathbf{A}) = 0$, then **A is not invertible.**

Matrix Inverse

We can also use the Normal Equations (see inverse methods):

$$\mathbf{G}^{-1} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T$$

such that:

$$\mathbf{m} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{d}$$