GEOS F493 / F693
Geodetic Methods and Modeling

– Lecture 03b: Position Estimations–

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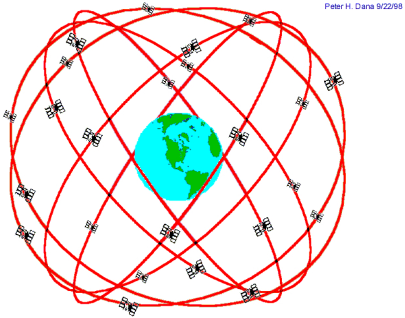
System Architecture

- Space Segment – satellites
- Control Segment – management of satellites
- User Segment – Civil and military receiver development

System Architecture: Space Segment

- Baseline constellation 24 satellites, 6 orbital planes, 55° inclined
- Period ≈ 12 hours, stationary ground tracks
- Currently 32 satellites operational



GPS Nominal Constellation
24 Satellites in 6 Orbital Planes
4 Satellites in each Plane
20,200 km Altitudes, 55 Degree Inclination

Signals

- continuous transmission on 2 L-band radio frequencies: Link 1 (L1), Link 2 (L2) (for legacy GPS)
- L1 (f_{L1} = 1575.42 MHz): 1 signal for civil users, 1 for military
- L2 (f_{L2} =1227.60 MHz): 1 signal military, new signals for civilian use (L2C, 19 satellites)
- L5 (1176.45 MHz): Safety of Life; civilian use (12 satellites)

Signals: Structure

- *Carrier:* sinusoidal signal with $f_{L1,2}$, derives from 10.23 MHz atomic clock

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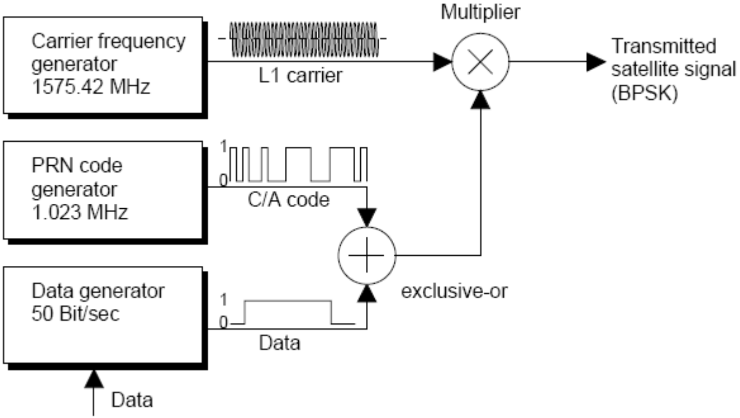
- *Ranging Code:* pseudo-random noise (PRN) sequences unique to satellite
- *Navigation Data:* satellite health, position, velocity, clock bias parameters, almanac (information/status on several/all satellites)

Signals: Structure



from: *http://www.ni.com/tutorial/7139/en/*

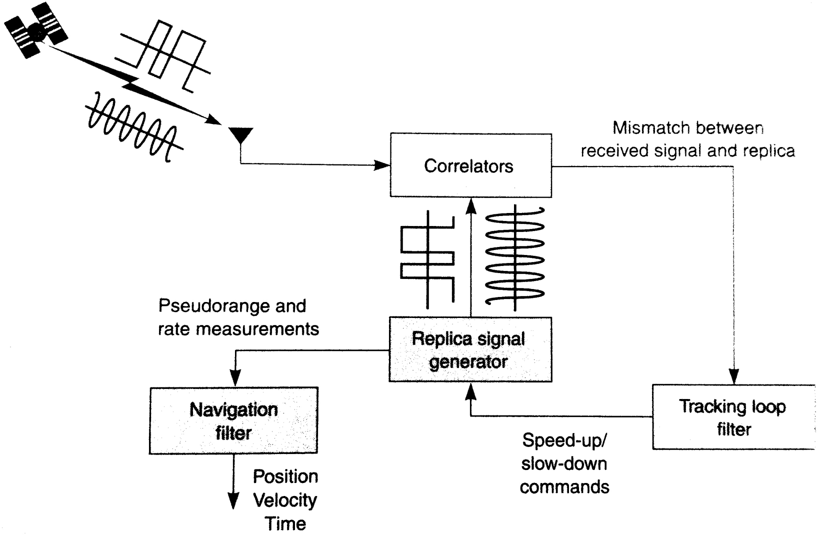
System Architecture: User Segment

Receivers: Signal Acquisition and Tracking

Receiver tasks:

- capture radio signals transmitted by satellites
- separate individual satellites
- measure signal transit time (crude)
- decode navigation message: gives satellite position, velocity, clock

Receivers



Misra and Enge, 2011, GPS–Signals, Measurements, and Performance

Coordinate System

Coordinate system in which user position is fixed:

- rotates with the Earth: conventional terrestrial reference system (CTRS)
- use Cartesian coordinate system
- define origin at center of mass
- z-axis is axis of rotation
- x-axis goes through intersection of equatorial plane and reference median
- y-axis makes it right-handed

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Easy, right?

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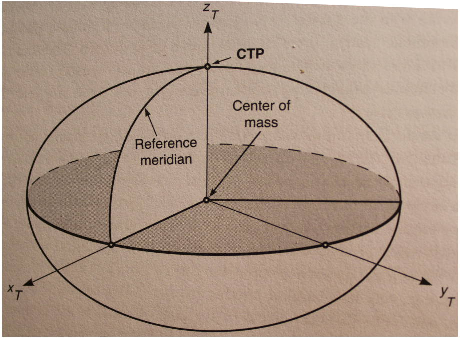

Misra and Enge, 2011, GPS–Signals, Measurements, and Performance

Coordinate System

What are potential issues?

Coordinate System

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- *Polar motion*: pole of rotation moves, roughly circular, several meters/year
- Use conventional terrestrial pole (CTP) – average of polar motion between 1900-1905
- *Center of mass*: where is it?

Coordinate System

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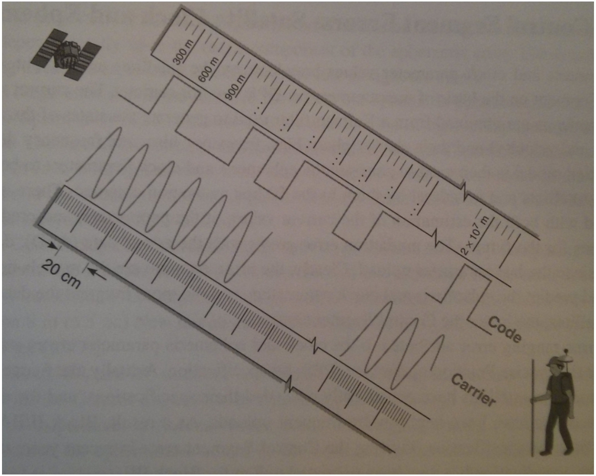
Actually,

- Need consistent coordinates from measurements (have errors!)
- *Realize* coordinate frame that fits measurements best (e.g, least-squares)
- World Geodetic System 1984 (WGS84) one such realization
- GPS positions in WGS84 ECEF coordinate frame
- Science: use various updates of ITRF (International Terrestrial Reference Frame)

Measurement Models

- Code Phase Measurement (today)
- Carrier Phase Measurement (next lecture)



Misra and Enge, 2011, GPS–Signals, Measurements, and Performance

Position Estimation w/ Pseudoranges

(Derivation also in notes)

- Positioning by (pseudo-)ranging
- Range: geometric distance between satellite and receiver
- Pseudorange: includes distance, clock error effects, path delays

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$$\rho^{(s)} = r^{(s)} + c(\delta t_u - \delta t^s) + I + T + \epsilon$$

$\rho^{(s)}$ - pseudorange to satellite s
$r^{(s)}$ - true range to satellite s
c - speed of light
δt_u - receiver clock bias
δt^s - satellite clock bias
I, T - Ionospheric and tropospheric delays
ϵ - unmodeled effects, measurement errors, etc.

Geometric Range

- We want to estimate receiver position
- Use Euclidean distance between receiver and satellite s:

- Receiver at position (x, y, z)
- Satellite, s, at position $(x^{(s)}, y^{(s)}, z^{(s)})$.
- Range $r^{(s)}$ is time dependent (not explicitly stated here)

Geometric Range

- We want to estimate receiver position
- Use Euclidean distance between receiver and satellite s:

$$r^{(s)} = \sqrt{(x^{(s)} - x)^2 + (y^{(s)} - y)^2 + (z^{(s)} - z)^2}$$

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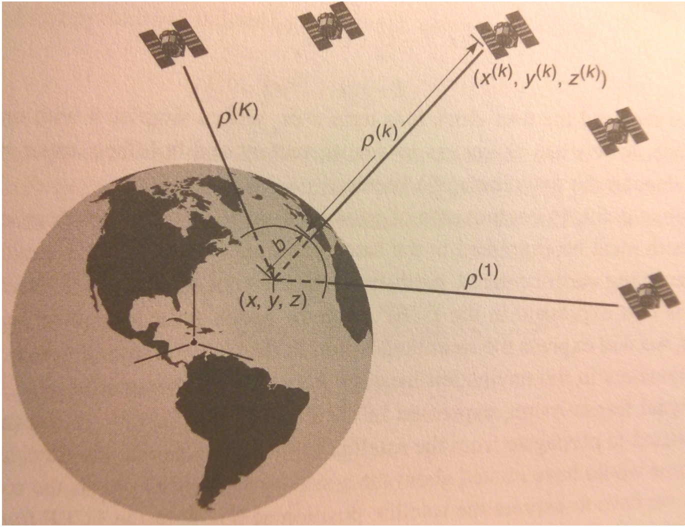
Geometric Range



Misra and Enge, 2011, GPS–Signals, Measurements, and Performance

Position Estimation w/ Pseudoranges

Insert geometric form of $r^{(s)}$ in Measurement Model:

$$\rho^{(s)} = \sqrt{(x^{(s)} - x)^2 + (y^{(s)} - y)^2 + (z^{(s)} - z)^2} + c\delta t_u - c\delta t^{(s)} + \epsilon$$

- ϵ now captures all errors incl. ionosphere, troposphere
- $x, y, z, \delta t_u$ are unknown, need to solve for those

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- What's the hangup?

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    - Non-linear in x, y, z

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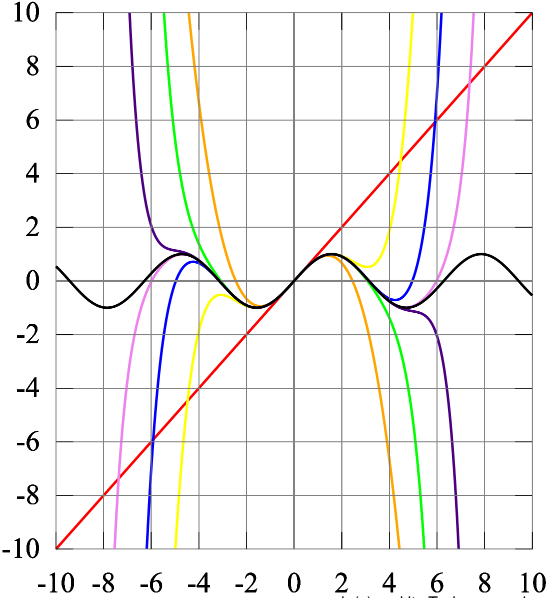
- ϵ now captures all errors incl. ionosphere, troposphere
- $x, y, z, \delta t_u$ are unknown, need to solve for those
- What's the hangup?
    - Non-linear in x, y, z ...try linear approximation

Taylor Expansion



sin(x) and its Taylor approximations; *source wikipedia*

Position Estimation w/ Pseudoranges

- Assume we can approximate $\rho^{(s)}$ with a linear function in the vicinity of a point.
- Use the linear parts of a **multivariate Taylor Series expansion** of $\rho^{(s)}$
- Linear approximation about point (a, b) for any function $f(x, y)$ (differentiable at least once) given by:

Position Estimation w/ Pseudoranges

- Assume we can approximate $\rho^{(s)}$ with a linear function in the vicinity of a point.
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- Linear approximation about point (a, b) for any function $f(x, y)$ (differentiable at least once) given by:

$$f(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

- Sum of the function value and its partial derivatives at (a, b)

Position Estimation w/ Pseudoranges

Linearizing $\rho^{(s)}$ about an approx. position and expected receiver clock bias (x_0, y_0, z_0, t_{e_0}) using multivariate Taylor Series expansion yields:

$$
\begin{aligned}
\rho^{(s)}(x, y, z, t_e) \quad = \quad & \rho^{(s)}(x_0, y_0, z_0, t_{e_0}) + \\
& \frac{\partial \rho^{(s)}}{\partial x}(x - x_0) + \frac{\partial \rho^{(s)}}{\partial y}(y - y_0) + \\
& \frac{\partial \rho^{(s)}}{\partial z}(z - z_0) + \frac{\partial \rho^{(s)}}{\partial t_e}(t_e - t_{e_0}) + \epsilon
\end{aligned}
$$

Position Estimation w/ Pseudoranges

We can simplify this:

$$\rho^{(s)}(x, y, z, t_e) - \rho^{(s)}(x_0, y_0, z_0, t_{e_0}) \quad = \quad \frac{\partial \rho^{(s)}}{\partial x}\Delta x + \frac{\partial \rho^{(s)}}{\partial y}\Delta y + \frac{\partial \rho^{(s)}}{\partial z}\Delta z + \frac{\partial \rho^{(s)}}{\partial t_e}\Delta t_e + \epsilon$$

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\Delta \rho^{(s)} &= \begin{bmatrix} \frac{\partial \rho^{(s)}}{\partial x} & \frac{\partial \rho^{(s)}}{\partial y} & \frac{\partial \rho^{(s)}}{\partial z} & \frac{\partial \rho^{(s)}}{\partial t_e} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_e \end{bmatrix} + \epsilon
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- We substituted δt_u with t_e to avoid double deltas.

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\end{aligned}$$

- We substituted δt_u with t_e to avoid double deltas.
- $\Delta \rho^{(s)}$ is the difference between measured pseudorange and expected geometric range between a given satellite position and the receiver apriori position.

Position Estimation w/ Pseudoranges

- Getting close to a solution!
- Need to calculate partial derivatives $\begin{bmatrix} \frac{\partial \rho^{(s)}}{\partial x} & \frac{\partial \rho^{(s)}}{\partial y} & \frac{\partial \rho^{(s)}}{\partial z} & \frac{\partial \rho^{(s)}}{\partial t_e} \end{bmatrix}$.

Position Estimation w/ Pseudoranges

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- Remember that we're using this measurement model:

$$\rho^{(s)} = \sqrt{(x^{(s)} - x)^2 + (y^{(s)} - y)^2 + (z^{(s)} - z)^2} + c\delta t_u - c\delta t^{(s)} + \epsilon$$

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- Remember that we're using this measurement model:

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- Let's work on this for the term $\frac{\partial \rho^{(s)}}{\partial x}$. We will need the chain rule:

$$\frac{\partial u^n}{\partial x} = n u^{n-1} \frac{\partial u}{\partial x}$$

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- Set u to be the term under the square-root in the range expression:

$$u = (x^{(s)} - x)^2 + (y^{(s)} - y)^2 + (z^{(s)} - z)^2$$

Position Estimation w/ Pseudoranges

We write:

$$\frac{\partial \rho^{(s)}}{\partial x} = \frac{\partial \sqrt{(x^{(s)} - x)^2 + (y^{(s)} - y)^2 + (z^{(s)} - z)^2}}{\partial x}$$

Position Estimation w/ Pseudoranges

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\begin{aligned}
\frac{\partial \rho^{(s)}}{\partial x} &= \frac{\partial \sqrt{(x^{(s)} - x)^2 + (y^{(s)} - y)^2 + (z^{(s)} - z)^2}}{\partial x} \\
&= \frac{\partial \sqrt{u}}{\partial x} = \frac{\partial u^{\frac{1}{2}}}{\partial x}
\end{aligned}
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&= \frac{1}{2u^{\frac{1}{2}}} \frac{\partial [(x^{(s)}-x)^2]}{\partial x} \\
&= \frac{2(x^{(s)}-x)}{2u^{\frac{1}{2}}} (-1)
\end{aligned}
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Position Estimation w/ Pseudoranges

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&= \frac{2(x^{(s)} - x)}{2u^{\frac{1}{2}}} (-1) \\
&= \frac{x - x^{(s)}}{\rho^{(s)}}
\end{aligned}
$$

Position Estimation w/ Pseudoranges

For all the partial derivatives at the apriori position:

$$\begin{aligned}
\frac{\partial \rho^{(s)}}{\partial x} &= \frac{x_0 - x^{(s)}}{\rho_0^{(s)}} \\
\frac{\partial \rho^{(s)}}{\partial y} &= \frac{y_0 - y^{(s)}}{\rho_0^{(s)}} \\
\frac{\partial \rho^{(s)}}{\partial z} &= \frac{z_0 - z^{(s)}}{\rho_0^{(s)}} \\
\frac{\partial \rho^{(s)}}{\partial t_e} &= c
\end{aligned}$$

- c is speed of light, this follows from earlier expressions of δt_u
- $\rho_0^{(s)}$ is geometric range from receiver apriori position to satellite s.

Position Estimation w/ Pseudoranges

With these expressions for the partial derivatives, we can write:

$$\Delta\rho^{(s)} = \left[\begin{array}{cccc} \frac{x_0 - x^{(s)}}{\rho_0^{(s)}} & \frac{y_0 - y^{(s)}}{\rho_0^{(s)}} & \frac{z_0 - z^{(s)}}{\rho_0^{(s)}} & c \end{array}\right] \left[\begin{array}{c} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_e \end{array}\right] + \epsilon$$

Position Estimation w/ Pseudoranges

With n satellites in view, we have n pseudorange measurements: $\rho^{(1)}, \ldots, \rho^{(n)}$ and can set up a linear system of equations:

$$\begin{bmatrix} \Delta\rho^{(1)} \\ \Delta\rho^{(2)} \\ \vdots \\ \Delta\rho^{(n)} \end{bmatrix} = \begin{bmatrix} \frac{x_0 - x^{(1)}}{\rho_0^{(1)}} & \frac{y_0 - y^{(1)}}{\rho_0^{(1)}} & \frac{z_0 - z^{(1)}}{\rho_0^{(1)}} & c \\ \frac{x_0 - x^{(2)}}{\rho_0^{(2)}} & \frac{y_0 - y^{(2)}}{\rho_0^{(2)}} & \frac{z_0 - z^{(2)}}{\rho_0^{(2)}} & c \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x_0 - x^{(n)}}{\rho_0^{(n)}} & \frac{y_0 - y^{(n)}}{\rho_0^{(n)}} & \frac{z_0 - z^{(n)}}{\rho_0^{(n)}} & c \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_e \end{bmatrix} + \epsilon$$

Solving the System (see Inverse Methods!)

System is of the form $Gm = d$

- G is the matrix with the partial derivatives
- d is the vector with the pseudorange differences
- m is the vector with the unknowns.

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We can introduce a weight matrix W, for instance, to put less emphasis on satellites at low elevation angles:

$$m = (G^T W G)^{-1} G^T W d$$

Solving the System (see Inverse Methods!)

Once we have a solution $m = [\Delta x, \Delta y, \Delta z, \Delta t_e]$ we can add these values to the apriori values to get an update:

$$\begin{bmatrix} x_{new} \\ y_{new} \\ z_{new} \\ t_{e_{new}} \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ t_{e_0} \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_e \end{bmatrix}$$

and iterate until improvements are small.

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In this week's lab you will implement this yourself!