ERTH 491-01 / GEOP 572-02
Geodetic Methods

– Lecture 20: Modeling - Strain –

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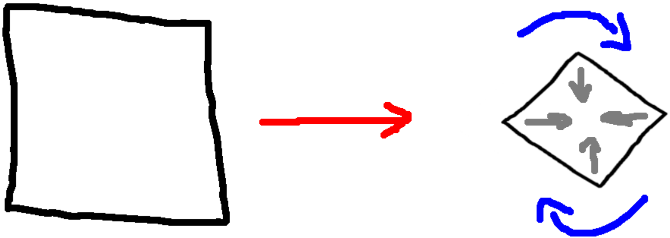
Background

- stresses deform solids
- small deformations: displacements much smaller than dimensions of the body
- hence, infinitesimal strain theory holds
- need finite strain theory when deformed body “much” different from undeformed body
- in geodesy, we get away with infinitesimal strains.
- deformation is a transformation

Transformations

- we've seen some of this before
- mathematical transformation maps something from one system / space into another system / space
- example: transform vector coordinates from one coordinate system to another coordinate system
- general form: $\mathbf{y} = A\mathbf{x}$, where x,y are vectors, A is transformation matrix
- linear transformation are part of basis of linear algebra
- hence, most common representations of linear transformation are **matrices** and **index notation**

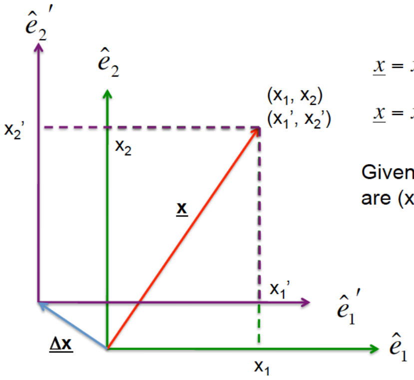
Deformation



deformation = translation + rotation + dilatation

- translation, rotation: rigid body deformation (angles, volume preserved)
- dilatation: volume changes, angles change

Transformations: Translation



$$\underline{x} = x_1\hat{e}_1 + x_2\hat{e}_2$$

$$\underline{x} = x_1'\hat{e}_1' + x_2'\hat{e}_2'$$

Given (x_1, x_2), what are (x_1', x_2')?

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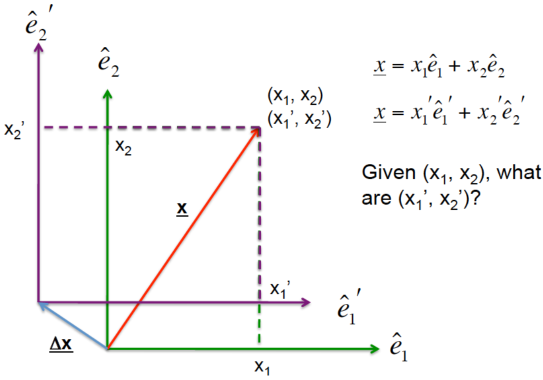
Transformations: Translation



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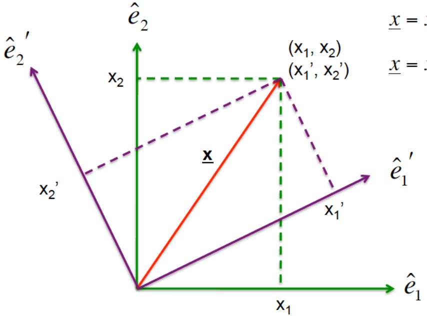
$$
\begin{aligned}
x_1' &= x_1 + \Delta x_1 \\
x_2' &= x_2 + \Delta x_2
\end{aligned}
$$

(indices indicate vector components!)

Transformations: Rotation



$$\underline{x} = x_1\hat{e}_1 + x_2\hat{e}_2$$

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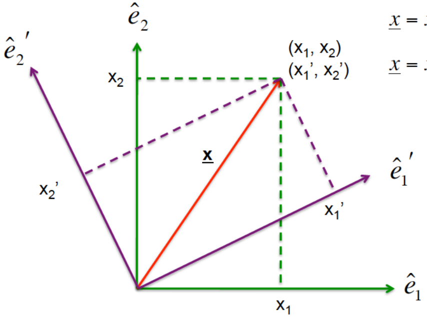
Transformations: Rotation



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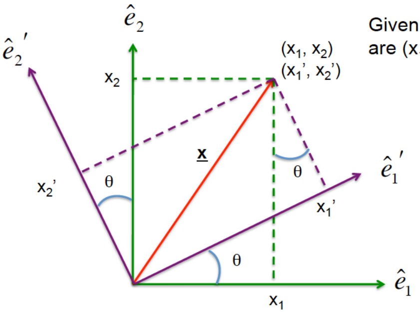
Given 2 systems, how are vector components related?

Transformations: Rotation



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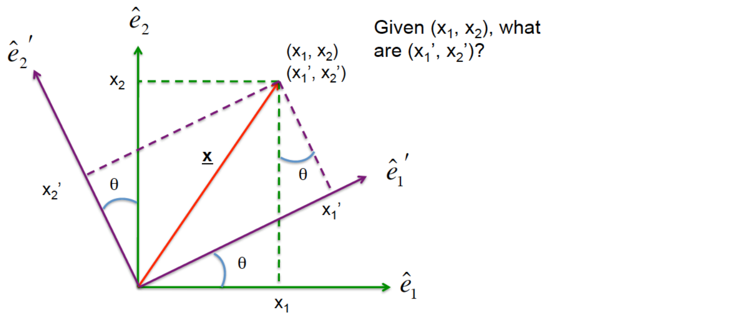
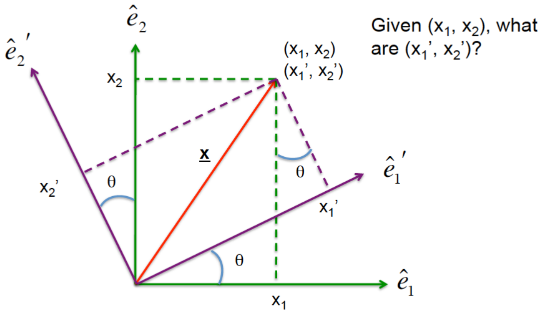
Transformations: Rotation



Given 2 systems, how are vector components related?

$$x_1 = x_1' cos(\theta) - x_2' sin(\theta) \qquad x_1' = x_1 cos(\theta) + x_2 sin(\theta)$$
$$x_2 = x_1' sin(\theta) + x_2' cos(\theta) \qquad x_2' = -x_1 sin(\theta) + x_2 cos(\theta)$$

Transformations: Rotation



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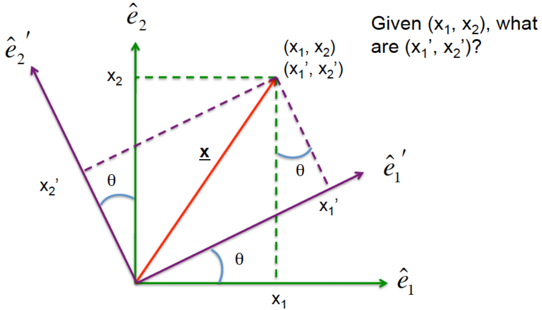
Given 2 systems, how are vector components related?

$$\mathbf{x} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \quad \mathbf{x}' = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Transformations: Rotation



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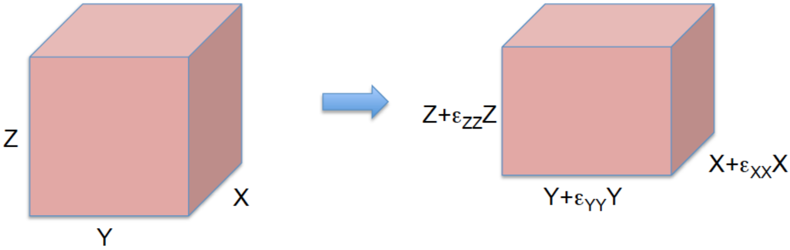
Rotating a vector is the same as rotating the coordinate system in the opposite direction

Transformations: Dilatation



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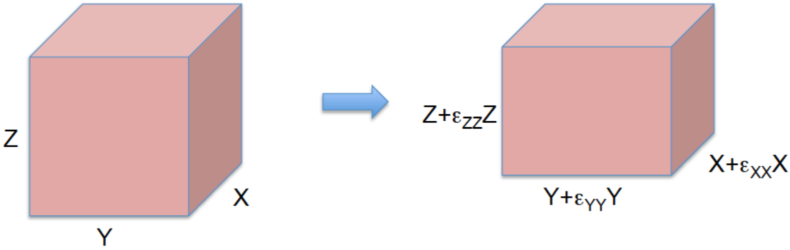
Transformations: Dilatation



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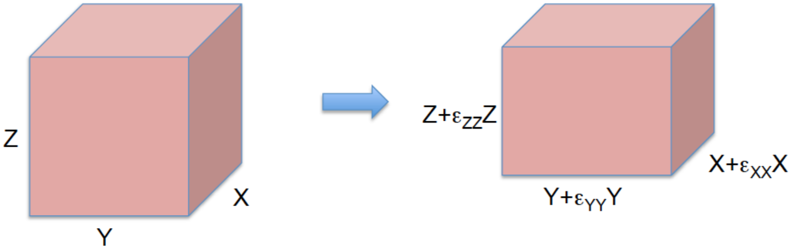
fractional length changes are **normal strains**:

$$\frac{change\_in\_length}{original\_length} = strain$$
$$du_1/X = \varepsilon_{xx}$$
$$du_2/Y = \varepsilon_{yy}$$
$$du_3/Z = \varepsilon_{zz}$$

convention important: geologists often use positive = contraction, can be extension, too. Check!

Transformations: Dilatation



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think in finite differences (infinitesimal lengths):

$$\lim_{length \to 0} \frac{length - new\_length}{length} = derivative$$

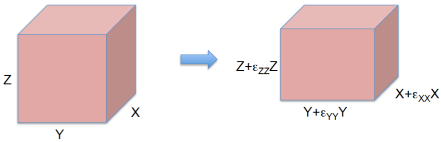
$$\partial u_1 / \partial x = \varepsilon_{xx}$$

$$\partial u_2 / \partial y = \varepsilon_{yy}$$

$$\partial u_3 / \partial z = \varepsilon_{zz}$$

convention important: geologists often use positive = contraction, can be extension, too. Check!

Transformations: Dilatation



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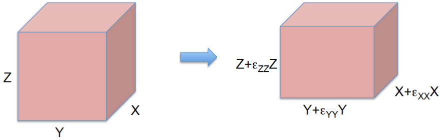
Dilatation (Δ) defined as fractional volume change:

$$\begin{aligned}\Delta &= \frac{X(1+\varepsilon_{xx}) * Y(1+\varepsilon_{yy}) * Z(1+\varepsilon_{zz}) - X * Y * Z}{X * Y * Z} \\ &= \frac{X * Y * Z((1+\varepsilon_{xx}) * (1+\varepsilon_{yy}) * (1+\varepsilon_{zz}) - 1)}{X * Y * Z} \\ &= (1+\varepsilon_{xx}) * (1+\varepsilon_{yy}) * (1+\varepsilon_{zz}) - 1\end{aligned}$$

Transformations: Dilatation



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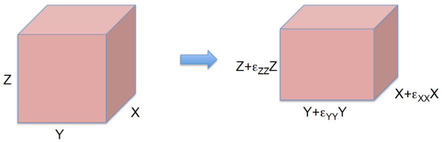
We use infinitesimal strain, products of strain can be dropped:

$$\begin{aligned}\Delta &= 1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} - 1 \\ &= \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\end{aligned}$$

Transformations: Dilatation



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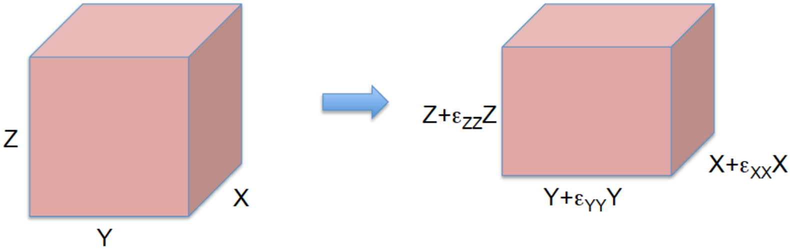
We use infinitesimal strain, products of strain can be dropped:

$$\begin{aligned}\Delta &= 1+\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}-1 \\ &= \varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}\end{aligned}$$

seismic P waves are travelling oscillations of Δ

Strain: Normal Strain



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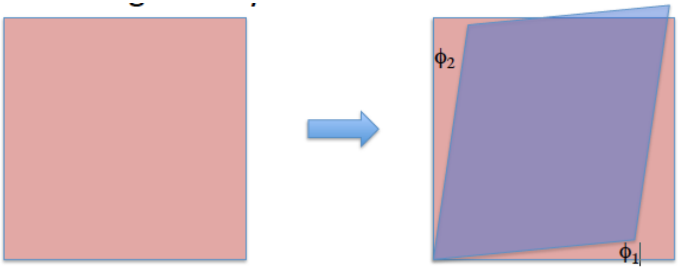
fractional length changes are **normal strains**:

$$\begin{aligned} \partial u_1/\partial x &= \varepsilon_{xx} \\ \partial u_2/\partial y &= \varepsilon_{yy} \\ \partial u_3/\partial z &= \varepsilon_{zz} \end{aligned}$$

components of strain proportional to derivatives of displacements in respective directions

Strain: Shear Strain



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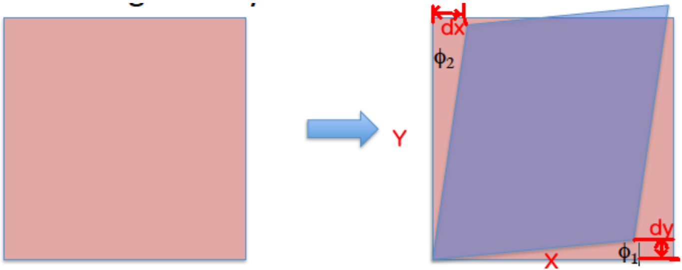
shear components of strain measure change in shape / angles

$$\varepsilon_{xy} = \varepsilon_{yx} = -\frac{1}{2}(\phi_1 + \phi_2)$$

Strain: Shear Strain



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shear components of strain measure change in shape / angles

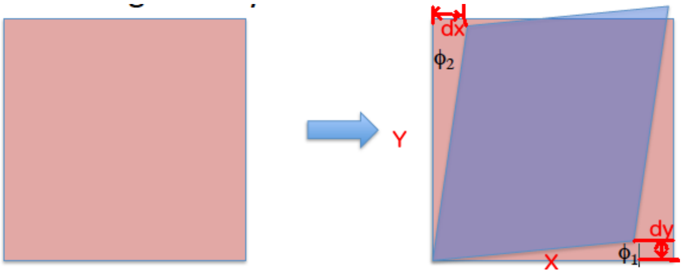
$$\varepsilon_{xy} = \varepsilon_{yx} = -\frac{1}{2}(\phi_1 + \phi_2)$$

angles are related to displacements:

$$tan(\phi_1) = \phi_1 = -\frac{dy}{X}$$

$$tan(\phi_2) = \phi_2 = -\frac{dx}{Y}$$

Strain: Shear Strain



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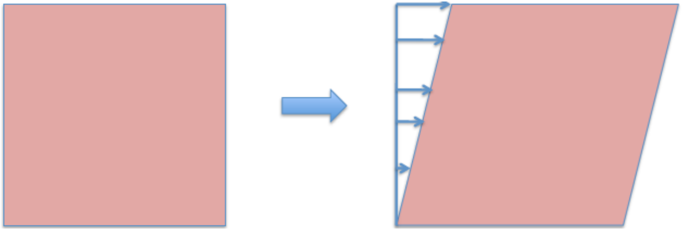
shear components of strain measure change in shape / angles

$$\varepsilon_{xy} \quad = \quad \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}\right)$$

subscripts: 1st – direction normal to element, 2nd – direction of shear

Strain: Shear Strain



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shear strain results in solid body rotation if $\phi_1 \neq \phi_2$:

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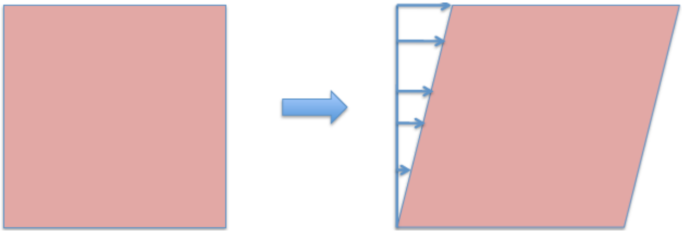
Strain: Shear Strain



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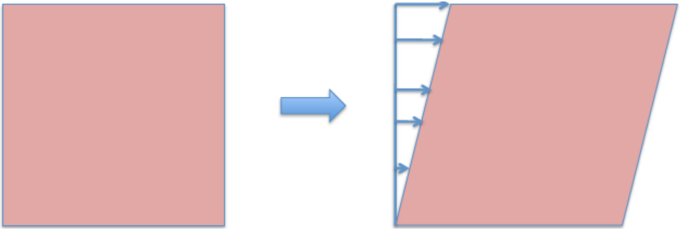
shear strain results in solid body rotation if $\phi_1 \neq \phi_2$:

$$\omega_z = -\frac{1}{2}(\phi_1 - \phi_2) = \frac{1}{2}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)$$



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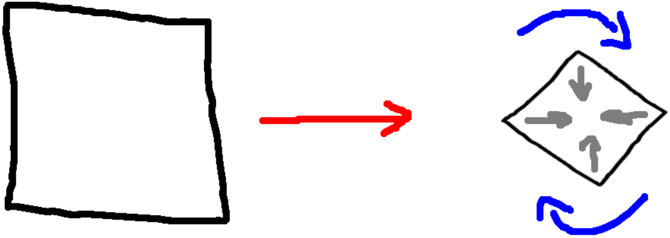
- if $\phi_1 = \phi_2$: no solid body rotation – **pure shear**
- if $\phi_1 = 0$: solid body rotation + shear – **simple shear** (strike slip faulting)

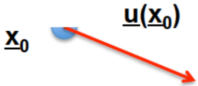
Putting it all together



$$\begin{aligned} \textit{deformation} &= \textit{translation} + \textit{dilatation} + \textit{rotation} \\ u &\approx x + dx + \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned}$$

correct formal description follows . . .

Deformation



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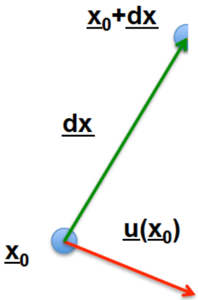
Deformation



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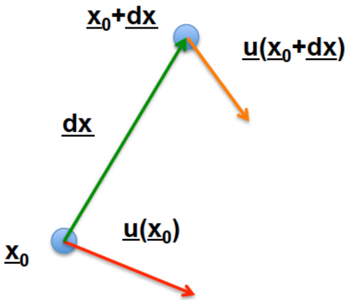
Deformation



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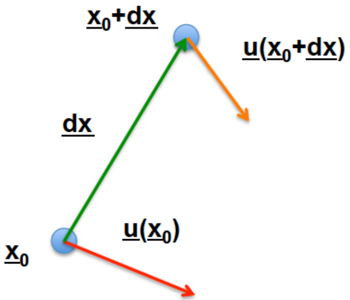
Deformation



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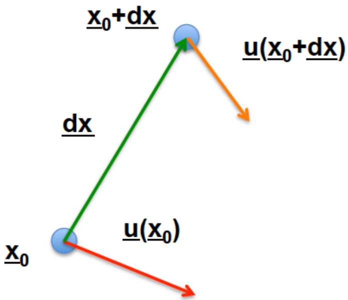
use Taylor Series expansion to relate the two vectors:

$$u_i(\mathbf{x_0} + \mathbf{dx}) \quad = \quad u_i(\mathbf{x_0}) + \left(\frac{\partial u_i}{\partial x_1}\right) dx_1 + \left(\frac{\partial u_i}{\partial x_2}\right) dx_2 + \left(\frac{\partial u_i}{\partial x_3}\right) dx_3$$

Deformation



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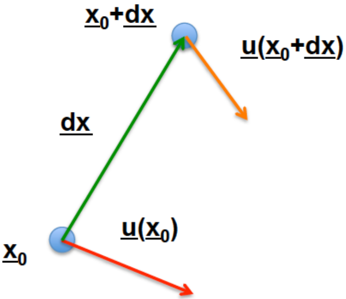
3 equations: i=1,2,3

first term: translation, remainder: rotation + dilatation

9 values $\partial u_i / \partial x_j$ for $i, j = 1 \ldots 3$

Deformation Tensor



$$u(\mathbf{x_0} + \mathbf{dx}) \quad = \quad u(\mathbf{x_0}) + \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

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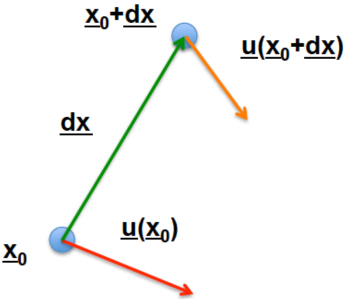
Deformation Tensor



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- matrix describes dilatation and rotation
- is a 2-direction (rank 2) tensor: contains normal strain, and strain perpendicular to face on which it acts
- think of tensors as extension of vectors (magnitude and direction), which are an extension of scalars (magnitude)

Separate Rotation and Strain

We can separate gradient tensor into the sum of two tensors: strain tensor and rotation tensor from:

$$\textit{deformation} = \textit{translation} + \textit{dilatation/strain} + \textit{rotation}$$

rotation is anti-symmetric (see rotation matrix), strain part is symmetric

Separate Rotation and Strain

We can separate gradient tensor into the sum of two tensors: strain tensor and rotation tensor from:

$$\begin{aligned} \textit{deformation} &= \textit{translation} + \textit{dilatation/strain} + \textit{rotation} \\ u_i(\mathbf{x_0} + \mathbf{dx}) &= u_i(\mathbf{x_0}) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) dx_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) dx_j \end{aligned}$$

rotation is anti-symmetric (see rotation matrix), strain part is symmetric

Separate Rotation and Strain

We can separate gradient tensor into the sum of two tensors: strain tensor and rotation tensor from:

$$
\begin{aligned}
\textit{deformation} &= \textit{translation} + \textit{dilatation/strain} + \textit{rotation} \\
u_i(\mathbf{x_0} + \mathbf{dx}) &= u_i(\mathbf{x_0}) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) dx_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) dx_j
\end{aligned}
$$

$$
\begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\
\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\
\frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}
\end{bmatrix}
=
\begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\
\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\
\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3}
\end{bmatrix}
+
$$

$$
\begin{bmatrix}
0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\
\frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\
\frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & 0
\end{bmatrix}
$$

rotation is anti-symmetric (see rotation matrix), strain part is symmetric